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FREENESS OF ORTHOGONAL MODULES*

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A finitely generated module M over a commutative ring with unit R is said to be orthogonal stably free of type (n, m) if M is isomorphic to the solution space of a $m \times n$ matrix α such that $\alpha\alpha^{\mathrm{t}} = I_m$. Geramita and Pullman have defined "generic" orthogonal stably free modules for each possible type and have obtained results on the freeness of these modules and on the supremum of the ranks of their free direct summands. We obtain further results of this type, concerning the generic modules of Geramita and Pullman as well as their sums with free modules and, in a few cases, their iterated sums. The last results are related to a theorem of T.Y. Lam stating that the iterated sum $r \cdot M$ of a stably free module M is free if r is greater than some lower bound. This lower bound is shown to be best possible in some cases.

Preliminaries and theorems

All rings considered in this paper are assumed to be commutative and to have an identity element. Modules are assumed to be finitely generated and unitary.

Given a ring R, a module M over R is said to be stably free of type (n, m) if $M \oplus R^m \simeq R^n$. A stably free module M is said to be orthogonal if $M \simeq \operatorname{Ker} \alpha$ where $\alpha\colon R^n \to R^m$ is an epimorphism such that $\alpha\alpha^{\mathrm{t}} = I$ (we identify α with the matrix representing α; α^{t} denotes the transpose of α and I denotes a suitable identity matrix).

The object of this paper is to study certain "generic" orthogonal stably free modules defined as follows. For $1 \leq m \leq n-1$, let

$$R_{n,m} = \mathbf{Z}[X_{11}, \ldots, X_{ij}, \ldots, X_{mn}]/\mathscr{I}_{n,m}$$

where X_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$ are indeterminates and $\mathscr{I}_{n,m}$ is the ideal of $\mathbf{Z}[X_{11}, \ldots, X_{ij}, \ldots, X_{mn}]$ generated by the entries of the matrix $[X_{ij}][X_{ij}]^{\mathrm{t}} - I_m$. Finally, let

$$P_{n,m} = \operatorname{Ker}\{\alpha : (R_{n,m})^n \to (R_{n,m})^m\}$$

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where $\alpha = \alpha_{n,m}$ is the homomorphism represented by the $m \times n$ matrix $[X_{ij}]$. (We use the same notation for the indeterminates and their canonical images in $R_{n,m}$.) Clearly, $\alpha\alpha^t = I$. Therefore, the modules $P_{n,m}$ are stably free of type (n, m) and orthogonal.

These definitions are special cases of the definitions found in [5]. Accordingly, our notation is simpler. In particular, the module $P_{n,m}$ defined above corresponds to the module $P^0_{m,n}(\mathbf{Z})$ of [5]. The reader should notice that we have changed the order of the indices involved, in order to follow a long established notation system for Stiefel manifolds (these manifolds are closely related to the modules $P_{n,m}$ and are instrumental in this paper).

Actually, our results are extensions of the results of [5], and the reader is referred to this paper for background information and motivation. We now proceed to state these results.

For any R-module M, define $\rho(M)$ to be the supremum of the ranks of the free direct summands of M. For any positive integer $k = 2^{4a+b}(2c+1)$, where $0 \leqslant b \leqslant 3$, let $\rho(k) = 8a + 2^b$.

Theorem 1. *Let $P_{n,m}$ be as above and let $1 \leqslant m \leqslant n-1$. Then*

(i) $\rho(P_{n,1}) = \rho(n) - 1$,

(ii) $\rho(P_{n,m}) = 0$ *for* $2 \leqslant m \leqslant n-2$ *and* $(n, m) \neq (7, 2)$ *or* $(8, 3)$,

(iii) $\rho(P_{7,2}) = 1$,

(iv) $\rho(P_{8,3}) \leqslant 1$,

(v) $P_{n,n-1}$ *is free for* $n \geqslant 2$.

This theorem should be compared with the corollaries to Proposition 2 and to Theorem 3 of [5]. Unfortunately, we have been unable to verify that $P_{8,3}$ admits a rank one direct summand, even though there is a good candidate (see remarks below).

Theorem 2. *Assume that $2 \leqslant m \leqslant n-2$ if n is odd and that $2 \leqslant m \leqslant n-3$ if n is even. Then:*

(i) $P_{n,m} \oplus (R_{n,m})^{m-1}$ *is free if and only if* $n = 4$ *or* 8.

(ii) *If n is even and $2 \leqslant m \leqslant \rho(n)$, then*

$$\rho(P_{n,m} \oplus (R_{n,m})^{m-1}) = \rho(n) - 1.$$

(iii) *If n and m are odd, then*

$$\rho(P_{n,m} \oplus (R_{n,m})^{m-1}) = m - 1.$$

This theorem should be compared with the corollary to Proposition 2 of [5]. One should notice also that further results concerning the modules $P_{m,n} \oplus (R_{m,n})^k$ for $1 \leqslant k \leqslant m-2$ can be deduced from Theorem 2 using elementary arguments.

The modules $P_{n,m}$ are "generic" for orthogonal stably free modules, as explained in [5, Proposition 1]. Therefore, we can interpret Theorems 1 and 2 as a (nearly complete) classification of the essentially different types of orthogonal stably free modules existing.

There is a close relationship between stably free modules and unimodular matrices. In particular, Theorem 2 allows us to supply examples of l-stable unimodular matrices for various values of l. (See [4] for the definition of l-stability.) The following corollary of Theorems 1 and 2 is obtained easily after applying Theorem 2.3 of [4].

Corollary 3. *For $1 \leq s \leq m-1$, let $\alpha_{n,m,s}$ be the $m \times (n+s)$ matrix, with entries in $R_{n,m}$, of the form $(\alpha_{n,m}, 0_s)$ where $\alpha_{n,m}$ is the matrix defined above and 0_s is the $m \times s$ 0-matrix. Assume that $1 \leq m \leq n-2$ if n is odd and that $1 \leq m \leq n-3$ if n is even. Also assume that $n \neq 4$ or 8. Then the matrix $\alpha_{n,m,s}$ is l-stable if and only if $l \leq s$.*

Finally, we would like to point out one more result. It has been shown in [6] that for any R-module M which is stably free of type (n, m), the r-fold direct sum $r \cdot M = M \oplus \cdots \oplus M$ (r-times) is free for $r \geq m + m/(n-m)$. We have the following result.

Theorem 4. *The module $r \cdot P_{n,m} = P_{n,m} \oplus \cdots \oplus P_{n,m}$ (r times) is not free if one of the following statements holds:*

(i) *$r = 2$, $m \geq 2$ and $n-m$ is odd and greater than or equal to 5.*

(ii) *r is odd, $3 \leq m \leq n-2$ and $r < (m+\omega)/(n-m)+1$ where $\omega = \omega(n, m)$ is 0 if m is even, -1 if m and n are odd, and 1 otherwise.*

Statement (i) shows that Theorem 2 of [6] is best possible for $m = 2$ when n is odd and greater than or equal to 7.

We will outline the proof of these theorems before making some remarks.

Proof of the theorems

Our theorems contain "negative" results, i.e. results giving an upper bound for $\rho(M)$ or stating that M is not free, for the various modules M involved, and "positive" results, i.e. results giving a lower bound on $\rho(M)$ or stating that M is free. Accordingly, the following outline of proof is divided into two parts.

Negative results

We will use the correspondence between isomorphism classes of real vector bundles over a compact Hausdorff space X and isomorphism classes of finitely generated projective modules over the ring $C(X)$ of continuous real valued functions defined on X. The following is a direct generalization of Example 1 of [7].

We denote points of $\mathbf{R}^n$ by column vectors. Let $\{e_i, i = 1, \ldots, n\}$ be the standard basis. Recall that the Stiefel manifold $V_{n,m}$ can be taken to be the set of real $m \times n$

matrices $x=(x_{ij})$ satisfying the equation $xx^t=I_m$. The trivial n-dimensional real vector bundle ε^n over $V_{n,m}$ consists of the pairs (x, u) where $x \in V_{n,m}$ and $u \in \mathbf{R}^n$. Let μ^m be the subbundle of ε^n consisting of the pairs (x, u) where $u = x^t\lambda$ for some $\lambda \in \mathbf{R}^m$. The m column vectors of the matrix x^t define a trivilization of μ^m. Consider the vector bundle map $f=f_{n,m}: \varepsilon^n \to \mu^m$ defined by $f(x, u)=(x, x^t x u)$. Then $\eta_{n,m} = \operatorname{Ker} f$ is a subbundle of ε^n.

We will identify the x_{ij}'s above with the coordinate functions on $V_{n,m}$. This allows us to obtain a natural inclusion of $R_{n,m}$ into $C(V_{n,m})$ defined by $X_{ij} \to x_{ij}$.

Now consider the $C(V_{n,m})$-module $\Gamma(\varepsilon^n)$ of continuous sections of ε^n. The sections $s_j : x \to (x, e_j)$ $(j=1, \ldots, n)$ form a basis of $\Gamma(\varepsilon^n)$. For $i=1, \ldots, m$, let $x_{i\cdot}$ be i-th column vector of the matrix x. Then the sections $t_i : x \to (x, x^t_{i\cdot})$ $(i=1, \ldots, m)$ form a basis of the free $C(V_{n,m})$-module $\Gamma(\mu^m)$. Relative to these bases, the induced homomorphism $\Gamma(f)$: $\Gamma(\varepsilon^n) \to \Gamma(\mu^m)$ is represented by the matrix $x=(x_{ij})$.

We now have a natural isomorphism

$$P_{n,m} \otimes_{R_{n,m}} C(V_{n,m}) \simeq \Gamma(\eta_{n,m}).$$

Indeed, the bases chosen above naturally define isomorphisms

$$(R_{n,m})^n \otimes C(V_{n,m}) \simeq \Gamma(\varepsilon^n)$$

and

$$(R_{n,m})^m \otimes C(V_{n,m}) \simeq \Gamma(\mu^m)$$

under which the homomorphism $\alpha_{n,m} \otimes 1$ is transformed into $\Gamma(f)$. (The tensor products are over $R_{n,m}$.) The above isomorphism follows immediately.

We now prove Theorem 4, a typical negative result. We must show that the module $r \cdot P_{n,m}$ is not free for the values of (n, m) given. It suffices to show that the module

$$\begin{aligned}(r \cdot P_{n,m}) \otimes C(V_{n,m}) &\simeq r \cdot (P_{n,m} \otimes C(V_{n,m}))\\ &\simeq r \cdot \Gamma(\eta_{n,m})\\ &\simeq \Gamma(r \cdot \eta_{n,m})\end{aligned}$$

is not free. Of course, the last module is free iff the vector bundle $r \cdot \eta_{n,m}$ is trivial [7]. That this vector bundle is not trivial for the values of n and m given is contained in the Theorems 1.2 and 1.3 of [2].

Other negative results are obtained in a similar way, calling upon the following results: for statement 1(i), a famous result of Adams (as pointed out in [4]); for 1(ii), 1(iii) and 1(iv), the theorems of [8] (a version of these theorems in the vocabulary used here can be found in [1]); for statement 2(i), Theorem 4.4 of [1]; for 2(ii), Theorem 4.1 of [1]; for 2(iii), Theorem 1.4 of [2].

Positive results

Most of the positive results stated are already known or are easy consequences of known results. In particular, statement 1(i) is the corollary of Theorem 3 of [5] and

1(v) is a well-known fact. Statement 1(ii) does not have a "positive" component. Statement 1(iii) however, does not seem to have appeared in the literature, at least in this form. The generator of a rank one direct summand of $P_{7,2}$ can be obtained explicitly by applying the formula for the cross-product in $\mathbf{R}^7$ to the "vectors" $(X_{11}, \ldots, X_{17})$ and $(X_{21}, \ldots, X_{27})$. One finds out that the resulting expression gives an element of $(R_{7,2})^7$. Further straightforward (but tedious) computation shows that this element actually lies in $P_{7,2} \subset (R_{7,2})^7$ and that it generates a direct summand.

As far as Theorem 2 is concerned, the positive part of 2(iii) is obvious. For 2(i) and 2(ii), the results follow from 1(i). Indeed, the module $P_{n,m} \oplus (R_{n,m})^{m-1}$ is orthogonal stably free of type $(n, 1)$. It follows from [5, proposition 1(3)] that there is an isomorphism

$$P_{n,m} \oplus (R_{n,m})^{m-1} \simeq P_{n,1} \otimes R_{n,m}$$

where the tensor product is taken over $R_{n,1}$. (The way in which $R_{n,m}$ is an $R_{n,1}$-module is not important here.) The results follow at once.

Remarks

(a) Robert Swift has indicated to us (December 1980) that he has shown that $\rho(P_{8,3}) = 1$ by explicitly describing a rank one free direct summand of $P_{8,3}$.

(b) As far as we know, it is not known whether or not $2 \cdot P_{n,2}$ is free for n even or equal to 5. The vector bundle $2 \cdot \eta_{n,2}$ is trivial in these cases however.

(c) Theorem 1.2 of [2] can be found also in [3].

(d) Algebraic proofs of the "negative" results would be more satisfying but appear difficult to obtain.

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